tion of the reasoning outlined above in the general three-dimensional case (or even for the case of three-dimensional perturbations of the flow here considered) leads to conclusions as follows:

1. For finite conductivity ($\eta \neq 0$) the equations are, as before evolutionary.

2. For perfect conductivity ($\eta = 0$) the characteristic equation generalizing (4) has part of its roots of the form $z = as + O(1)$, i.e. $\omega = as^2 + O(s)$ in which $a$ is real. The predominant term of this expression is "neutral", and the fulfillment of condition (5) for $s \to \infty$ depends on the next following term, while in the method of plane waves, which presupposes constant coefficients, the predominant asymptotic terms only have a meaning.

Thus, in the general nondissipative case the problem is much more involved, necessitating the consideration of variable coefficients of equations. In a way it will be a generalization of the system of equations of the complicated problem of the evolutionary property of the Schroedinger equation with variable coefficients.

## BIBLIOGRAPHY


1. Brushlinskii, K. V., Gerlakh, N. I. and Morozov, A. I., Computation of nonstationary two-dimensional flows of plasma of finite conductivity in the presence of the Hall effect. Magnetohydrodynamics, №1, 1967.
2. Kovrov, P. E., Morozov, A. I., Tokarev, L. G. and Shchepkin, G. Ia., Magnetic field distribution in a coaxial plasma injector. Dokl. Akad. Nauk SSSR, Vol. 172, №6, 1967.
3. Gel'fand, I. M., Certain problems of the theory of quasi-linear equations. Usp. Matem. Nauk, Vol. 14, №2, 1959.


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# SIMPLE WAVE IN A CONDUCTIVE MEDIUM IN A STATIONARY GRAVITATIONAL FIELD

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**1.** We shall consider the simultaneous nonstationary motion of a perfectly conductive gas in a homogeneous magnetic field $H$ perpendicular to the direction of velocity taking into account the force of gravity $g$. If at the initial instant the motion is isentropic, then with the condition of frozen lines of magnetic field $H = b\rho$ the coefficient $b$ will remain constant throughout the duration of motion.

Then, in the case here considered the Euler equation together with the continuity equation is reduced to the system of conventional gas dynamics equations [1 and 2]

$$u_t + uu_x + \frac{p_m}{\rho} = -g, \qquad \rho_t + u\rho_x + \rho u_x = 0, \qquad p_m = A\rho^k + \frac{H^2}{8\pi} \qquad (1.1)$$

The effective velocity of sound in this case is

$$c_m^2 = \left(\frac{\partial p_m}{\partial \rho}\right)_s = c^2 + v_a^2, \qquad c = (Ak\rho^{k-1})^{1/2}, \qquad v_a = \left(\frac{b^2\rho}{4\pi}\right)^{1/2} \qquad (1.2)$$

Here $c$ is the usual velocity of sound in a conductive medium, and $v_a$ the Alfven velocity.

Let $i_m$ be the effective enthalpy of the conductive gas. Then with the use of relationship $\rho^{-1}dP_m = c_m^2\rho^{-1}d\rho = di_m$ we can derive from system (1.1)

$$\left[\frac{\partial}{\partial t} + (u \pm c_m)\frac{\partial}{\partial x}\right] U(t) = 0, \qquad U(t) = u + gt \pm \int c_m \frac{d\rho}{\rho} \tag{1.3}$$

It will be seen from this that the specified state of the medium as defined by $U(t)$ spreads up-, and downstream of the flow at velocities $u \pm c_m$ while interacting with each other. The equilibrium of the conductive gas compressed by the gravitational field is expressed for $t \leqslant 0$ by Eq. $\frac{k}{k-1}A\rho^{k-1} + \frac{b^2\rho}{4\pi} = -gx + \text{const}$

and for the simple case of $k = 3$ the effective velocity of sound is defined by Formula

$$c_m^2 = 3A\left(B^2 - \frac{1}{4}b_0^4\right), \qquad B = -\frac{1}{2}b_0^2 + \left[b_0^2B(c_{m0}) + \frac{1}{2}b_0^4 + \frac{c_{m0}^2}{3A} - \frac{2gx}{3A}\right]^{1/2}$$

$$B(c_{m0}) = \left(\frac{1}{4}b_0^4 + \frac{c_{m0}^2}{3A}\right)^{1/2}, \qquad b_0^2 = \frac{b^2}{12\pi A} \tag{1.4}$$

Here $c_{m0}$ corresponds to the value of $c_m$ at $x = 0$.

At the initial instant of motion there exists an effective initial pressure distribution, and by the same token an effective velocity of sound to which corresponds Eq. (1.4) must also exist in the conductive gas column. Hence, the discharge of a conductive gas in a gravitational field cannot be defined by a particular solution of the system of Eqs. (1.1), i.e. it is necessary in this case to resort to a general solution of this system [2 and 3].

As the conditions $dx / dt = u \pm c_m$ define the characteristics of systems (1.1), hence along these lines the relationship

$$u + gt = \pm \int c_m \frac{d\rho}{\rho} + \text{const}$$

must be satisfied, or in the case here considered

$$u + gt = \pm \{c_m + b_0^2(3A)^{1/2}\ln[(B(c_m) + {}^1/_2b_0^2)^{1/2} + (B(c_m) - {}^1/_2b_0^2)^{1/2}]\} + C \tag{1.5}$$

Here, $C = \text{const}$ is defined by the initial condition.

When the discharge is to the left, the moving rarefaction wave front bordering at any given instant on the unperturbed area is defined by Eq.

$$dx / dt = -c_m = -[3A(B^2 - {}^1/_4b_0^4)]^{1/2}$$

Integrating this equation with condition $t = 0$, $B = B_0$ we obtain the law of motion of the front in the form

$$x = \frac{1}{2g}\left[c_m^2 - c_{m0}^2 + 3Ab_0^2\,(B(c_m) - B(c_{m0}))\right]$$

$$t = \frac{1}{g}\left[c_m - c_{m0} + \frac{1}{2}b_0^3(3A)^{1/2}\ln\frac{c_m + (c_m^2 + {}^3/_4Ab_0^4)^{1/2}}{c_{m0} + (c_{m0}^2 + {}^3/_4Ab_0^4)^{1/2}}\right]$$

At the rarefaction wave front $u = 0$, hence from (1.5) we have

$$gt = c_m - c_{m0} + b_0^2(3A)^{1/2}\ln\frac{(B(c_m) + {}^1/_2b_0^2)^{1/2} + (B(c_m) - {}^1/_2b_0^2)^{1/2}}{(B(c_{m0}) + {}^1/_2b_0^2)^{1/2} + (B(c_{m0}) - {}^1/_2b_0^2)^{1/2}}$$

The front of a conductive gas discharged into vacuum in which the magnetic field is absent attains the maximum expansion velocity, i.e. is subject to law

$$c_m = 0, \qquad u + gt = f(c_{m0})$$

$$f(c_{m0}) = c_{m0} + b_0^2 (3A)^{1/2} \{\ln [(B(c_{m0}) + {}^1/_2 b_0^2)^{1/2} + (B(c_{m0}) - {}^1/_2 b_0^2)^{1/2}] - \ln b_0\}$$

or to its equivalent

$$dx / dt = u = f(c_{m0}) - gt$$

Hence

$$x = tf(c_{m0}) - {}^1/_2 gt^2$$

The gas attains its maximum lift $x_{\max} = f^2(c_{m0}) / 2g$ for $t = f(c_{m0}) / g$, after which it begins to fall "down". A new wave is then generated which can be readily defined, as we have for it conditions $u = 0$ and $c_m^2 = 3A (B^2 - {}^1/_4 b_0^4)$

We note the following aspects. In order to simplify computations we may by a suitable selection of coefficients $b_1$ and $b_2$, and without appreciable loss of the result accuracy, substitute for the condition of frozen lines of the magnetic field $H = b\rho$ the expression

$$H^2 = b^2\rho^2 = b_1\rho^3 + b_2$$

The system of Eqs. (1.3) will then have the simple form

$$\frac{\partial}{\partial t}(u \pm c_m) + (u \pm c_m)\frac{\partial}{\partial x}(u \pm c_m) + g = 0$$

the solution of which obviously is

$$x = (u \pm c_m)\, t + {}^1/_2 gt^2 + F_{1,2}(u \pm c_m + gt)$$

Here $F_1(u + c_m + gt)$, $F_2(u - c_m + gt)$ are arbitrary functions. It may be said that in this case the two magnitudes $u \pm c_m$ propagate independently of each other in the form of two noninteracting simple waves.

If the discharge of the conductive gas which initially was in adiabatic equilibrium $c_m^2 = c_{m0}^2 - 2gx$, $u = 0$ is in the direction of positive values of $x$, then

$$x = ut + \frac{gt^2}{2} - \frac{1}{4g}(u + gt + c_m + c_{m0})^2 - 2c_{m0}(u + gt + c_m + c_{m0})]$$

Hence the motion of the rarefaction wave front is now in accordance with the law $x = -(c_{m0}t + {}^1/_2 gt^2)$, and that of the expansion front with

$$x = c_{m0}t - {}^1/_2 gt^2 \qquad (x_{\max} = c_{m0}^2 / 2g \text{ for } t = c_{m0} / g)$$

**2.** The motion of an incompressible perfectly conductive liquid flowing in an open channel in a transverse field may be represented by the analogous motion of a one-dimensional perfectly conductive gas. In this case it is necessary to assume that the depth of liquid and the channel width are small in comparison with the wave length [4]. The fundamental equations of such a flow are

$$u_t + uu_x + \rho^{-1}(P + H^2 / 8\pi)_x = 0, \qquad F_t + (uF)_x = 0, \qquad H = b_0 F \qquad (2.1)$$

where $F$ is the area of the channel cross section.

Since $dp = g\rho dh$, where $h$ is the channel depth, hence the equation of motion is of the form [2]

$$u_t + uu_x + gh_x + (H^2 / 8\pi\rho)_x = 0 \qquad (2.2)$$

In the case of a prismatic channel $F = F(h)$, therefore Eq. (2.2) with the aid of equality $gdh / dF = b_0 F$ may be written in the form

$$u_t + uu_x + GFF_x = 0, \qquad G = b_0 + b^2 / 4\pi\rho = \text{const}$$

By the introduction of the new variable $C_m^2 = CF^2$ we reduce system (2.1) to the convenient form [2 and 5]

$$u_t + uu_x + c_m c_{mx} = 0, \qquad c_{mt} + uc_{mx} + c_m u_x = 0 \qquad (2.3)$$

which in turn may be written in the form

$$\left[\frac{\partial}{\partial t} + (u \pm c_m)\frac{\partial}{\partial x}\right](u \pm c_m) = 0 \qquad (2.4)$$

Here $c_m$ is the effective velocity of the sound defined by (1.2), and in the case of a conductive liquid we have $c = \sqrt{2gh}$, $v_a = b\,(gh / 2\pi\rho b_0)^{1/2}$. It will be seen from system (2.4) that magnitudes $u \pm c_m$ have constant values for points moving in the conductive liquid at velocities $u \pm c_m$, i.e. for points the motions of which are defined by Eq. $dx / dt = u \pm c_m$, while the related perturbations moving towards each other do not interact between themselves.

Thus, system (2.4) coincides with the differential equations of the adiabatic flow of a perfect gas with the adiabatic exponent $k = 3$. This feature makes possible the direct application to this problem of all of the gas-dynamical results related to motions free of shock wave generation.

BIBLIOGRAPHY

1. Kaplan, S. A. and Staniukovich, K. P., Solution of equations of magnetogasdynamics for one-dimensional motion. Dokl. Akad. Nauk SSSR, Vol. 95, №4, 1954.
2. Staniukovich, K. P., Unsteady Motions of a Continuous Medium. M., Gostekhizdat, 1955.
3. Staniukovich, K. P. and Dzhusupov, K., Nonstationary discharge of gas in a constant gravitational field. Dokl. Akad. Nauk SSSR, Vol. 177, №4, 1967.
4. Khristianovich, S. A., Mikhlin, S. G. and Devison, B. B., Certain New Problems of Continuous Media Mechanics. Pt. 1, Ch. III, M.-L., Izd. Akad. Nauk SSSR, 1938.
5. Golitsyn, G. S., One-dimensional motions in magnetohydrodynamics. ZhETF, Vol. 35, №3, 1958.

Translated by J. J. D.

# STABILIZATION OF A NONLINEAR CONTROL SYSTEM IN THE CRITICAL CASE OF A PAIR OF PURELY IMAGINARY ROOTS

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We consider the problem of stabilization of the steady motions of a nonlinear control system in the critical case of a pair of purely imaginary roots. We introduce a nonanalytic control in two critical variables and use the Liapunov's classical theory of stability of motion [1 and 2] together with the methods developed in [3].

**1.** Let us consider the controlled system

$$\frac{dx}{dt} = Ax + Bu + g\,(x, u) \qquad (1.1)$$

where $x$ denotes the $(n + 2)$-dimensional perturbation vector, $u$ is the $m$-dimensional control vector which we shall assume to be unaffected by any disturbances, $A$ and $B$ are constant $(n + 2) \times (n + 2)$ and $(n + 2) \times m$ matrices, respectively, and $g\,(x, u)$